Bezout’s Theorem: Some Probabilistic Properties of System of Real and Complex Polynomial Equations

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Abstract. In this paper we study the geometric measure theory to estimate probability distribution of system of real or complex polynomial equations.

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1. Introduction

For \((d) = (d_1,\ldots,d_n)\), each \(d_i\) is a positive integer. Let \(P_{(d)}(n)\) be the space of systems of \(n\) real or complex polynomial equations in \(n\) variables \(f : \mathbb{K}^n \to \mathbb{K}^n\),
\[
f(x) = (f_1(x),\ldots,f_n(x)) = 0, \ x \in \mathbb{K}^n.
\]
We denote \(\Sigma \subset P_{(d)}(n)\) be the subset of singular systems then by Sard’s theorem \(\Sigma\) has zero Lebesgue measure.

The problem is: How are the “near” singular systems?
This problem is usually considered in the complexity theory and the numerical analysis. The main goal of this paper is to find out the following problems:

(i) About the condition number
(ii) To estimate the probability distribution of the condition number
(iii) To estimate the average loss of precision
(iv) To evaluate the average number of real roots of a real polynomial system and give some applications

To perform this work, we base mainly on some results of group L.Blum, F.Shub and S.Smalle (see [1], [7], [8], [9], [10], [11], [12]).

2. Linear case

Definition 2.1 Let \(A \in \mathbb{M}_n^k\) be an invertible real or complex \(n \times n\) matrix. The condition number of \(A\) is defined as \(\kappa(A) = \|A\|\|A^{-1}\|\), where \(\| \|\) is the operator norm.
**Remark 2.1**  The condition number measures the relative error in the solution of the system of linear equation $Ax = b$.

Let $\Sigma_n = \{ A \in \mathbb{M}_n^\times \mid \det A = 0 \}$ is the subset of singular matrices. We define

$$d_F(A, \Sigma_n) = \min_{M \in \Sigma_n} \| A - M \|_F$$

is the distance from $A$ to $\Sigma_n$, here $\| A \|_F$ denotes the Frobenius norm.

The following Eckart-Young theorem proves relating the condition number of a problem to its distance to the set of singular matrices $\Sigma_n$.

**Theorem 2.1 (Eckart-Young)**  Let $A \in \mathbb{M}_n^\times$ be an invertible matrix; then

$$d_F(A, \Sigma_n) = \frac{1}{\| A^{-1} \|}$$

The proof of this theorem may be found in BCSS(1997) (see[1]).

The next theorem is an estimate of the probability distribution of the condition number for the standard Gaussian probability distribution (see[1]).

**Theorem 2.2** We have

\[
\text{Prob}\left\{ A \in \mathbb{M}_n^\times \mid \kappa(A) > \frac{1}{\varepsilon} \right\} \leq \varepsilon n^{\frac{3}{2}} \quad \text{and}
\]

\[
\text{Prob}\left\{ A \in \mathbb{M}_n^\times \mid \kappa(A) > \frac{1}{\varepsilon} \right\} \leq \varepsilon n^4
\]

for the standard Gaussian probability distribution on $\mathbb{M}_n^\times$.

We close the linear case by estimating the average loss of precision for random matrices whose entries are independently distributed with respect to the standard Gaussian distribution.

**Theorem 2.3** For $b \in \mathbb{N}$ and $A \in \mathbb{M}_n^\times$, we have

$$\mathbb{E}\left\{ \log_b (\kappa(A)) \right\} \leq \frac{5}{2} \log_b n + \frac{1}{\ln b} \quad \text{and}
$$

for $b \in \mathbb{N}$ and $A \in \mathbb{M}_n^\times$, we have

$$\mathbb{E}\left\{ \log_b \kappa(A) \right\} \leq 2 \log_b n + \frac{1}{2 \ln b}$$

with the standard Gaussian probability distribution on $\mathbb{M}_n^\times$, here $\log_b \kappa(A)$ measures the loss of precision.
\textbf{a. Nonlinear case}

We now give a more general context for the condition number in nonlinear problems.

\textbf{Definition 3.1} Let $\mathcal{H}_{(d)}$ be the space of homogeneous polynomial mappings in $n+1$ variables of degree $(d)$, $f = (f_1, \ldots, f_n) : \mathbb{C}^{n+1} \to \mathbb{C}^n$. We define the solution variety as

$$V = \left\{(f, x) \in \mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}\left(\mathbb{C}^{n+1}\right) \mid f(x) = 0\right\}$$

then we can prove the solution variety $V$ is a smooth connected subvariety of $\mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ of codimension $n$.

We denote

$$\pi_1 : V \to \mathbb{P}\left(\mathcal{H}_{(d)}\right) \text{ and } \pi_2 : V \to \mathbb{P}\left(\mathbb{C}^{n+1}\right)$$

be the restriction to $V$ of the projection from $\mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ onto the first and second factors respectively.

\textbf{Definition 3.2} Let $\Sigma^' \subseteq V$ be the set of points where derivative of $\pi_1$ is singular and called the critical variety. The image $\pi_1(\Sigma^') = \Sigma$ is called the discriminat variety.

Consider the context of the implicit function theorem, let

$$F : \mathbb{P}\left(\mathcal{H}_{(d)}\right) \times \mathbb{P}\left(\mathbb{C}^{n+1}\right) \to \mathbb{P}\left(\mathbb{C}^{n+1}\right) \text{ be a } C^1 \text{ map, } F(f_0, \xi_0) = 0,$$

$$\frac{\partial F}{\partial \xi}(f_0, \xi_0) : \mathbb{P}\left(\mathbb{C}^{n+1}\right) \to \mathbb{P}\left(\mathbb{C}^{n+1}\right) \text{ non-singular.}$$

Then there exists a $C^1$ map $G : U \to \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ such that $G(f_0) = \xi_0$ and $F\left(f, G\left(f\right)\right) = 0$

$f \in U$. Here $U$ is a neighborhood of $f_0$ in $\mathbb{P}\left(\mathcal{H}_{(d)}\right)$.

We see that $F_j : \mathbb{P}\left(\mathbb{C}^{n+1}\right) \to \mathbb{P}\left(\mathbb{C}^{n+1}\right), F_j\left(\xi\right) = F\left(f, \xi\right)$, as a system of equations parameterized by $f \in \mathbb{P}\left(\mathcal{H}_{(d)}\right)$ then $f$ may be the input of a problem $F_j\left(\xi\right) = 0$ with output $\xi$. Let us call the derivative $DG : \mathbb{P}\left(\mathcal{H}_{(d)}\right) \to \mathbb{P}\left(\mathbb{C}^{n+1}\right)$ the condition matrix at $(f_0, \xi_0)$. If $(f_0, \xi_0) \in \Sigma^'$ then the condition number is defined by

$$\mu(f_0, \xi_0) = \|DG\left(f_0\right)\|, \text{ operator norm.}$$

Thus $\mu(f_0, \xi_0)$ is the bound on the infinitesimal output error of the system $F_j\left(\xi\right) = 0$ in terms of the infinitesimal input error.
The following condition number theorem gives an exact formula relating the condition number to the reciprocal of a distance to a set of singular systems.

**Condition Number Theorem.** Let \( f \in \mathbb{P}\left( \mathcal{F}_{(d)} \right), \varsigma \in \mathbb{P}\left( \mathbb{C}^{n+1} \right), f(\varsigma) = 0 \). Then

\[
\mu_{\text{norm}}(f, \varsigma) = \frac{1}{d_{\mathbb{P}(\mathcal{F}_{(d)})}(f, \varsigma), \Sigma \cap V_{\varsigma}}
\]

where \( d_{\mathbb{P}(\mathcal{F}_{(d)})} \) is the projective metric on \( \mathbb{P}(\mathcal{F}_{(d)}) \) and \( V_{\varsigma} = \left\{ f \in \mathbb{P}(\mathcal{F}_{(d)}) \mid f(\varsigma) = 0 \right\} \)

Then \( \mu_{\text{norm}}(f) = \max_{\varsigma \mid \mathbb{P}(\mathcal{F}_{(d)})} \mu_{\text{norm}}(f, \varsigma) = \frac{1}{\min_{\varsigma \mid \mathbb{P}(\mathcal{F}_{(d)})} d_{\mathbb{P}(\mathcal{F}_{(d)})}(f, \varsigma), \Sigma \cap V_{\varsigma}} \)

The proof uses unitary invariance of all the objects. Thus one can reduce to the point \( e = (1,0,...,0) \) and to the linear terms then apply the Eckart-Young theorem. For more detail, the proof may be found in BCSS (see[1]) or BS (See[8]).

The following formula is the basis result of geometric integration theory. This is a general case of Fubini's theorem.

**Co-area Formula** Let \( U \subset V \) be an open set, then

\[
\int_{x \in \pi_i^{-1}(U)} \#(\pi_i^{-1}(x) \cap U) = \int_{x \in \mathbb{P}(\mathbb{C}^{n+1})} \int_{x \in \pi_i^{-1}(x) \cap U} \text{det}(DG(a)DG(a)^*)^{-\frac{1}{2}}
\]

Here \( DG(a) \) is the condition matrix, \( DG(a)^* \) is its adjoint and \# means cardinality.

Next we apply this formula to estimate the probability distribution of the condition number for nonlinear problems.

**Theorem 3.1** For \( n > 1 \) then

\[
\text{Prob}\left\{ f \in \mathbb{P}(\mathcal{F}_{(d)}) \mid \mu_{\text{norm}}(f) > \frac{1}{\varepsilon} \right\} \leq \varepsilon^s n^s (n+1)(N-1)(N-2) \mathcal{D}
\]

where \( \varepsilon > 0, f \in \mathbb{P}(\mathcal{F}_{(d)}), N = \text{dim} \mathcal{F}_{(d)}, \mathcal{D} = \prod_{i=1}^{n} d_i \) is the Bezout number.

3. **The average number of real roots**

We consider the system of \( n \) polynomial equations in \( n \) variables

\[
f = (f_1, ..., f_n) = 0 \quad (4.1)
\]

where, \( f_i \) is the polynomial with complex coefficients of degree \( d_i, i = 1...n \).

**Bezout Theorem** The system (4.1) has \( \mathcal{D} = d_1 \times \cdots \times d_n \) non-degenerate complex roots.
Problem is: *How are the number of real roots?*

In this section we apply the co-area formula of geometric integration theory to evaluate the average number of real roots and give some applications.

Let $S^R_{(d)} \subset S_{(d)}$ be the subspace of systems of equations with real coefficients.

**Definition 4.1** For $f \in \mathbb{P}\left(S^R_{(d)}\right)$, let $(f)$ denote the number of real roots of $f$. Then we call

$$A_{(d)} = \frac{1}{\text{Vol}\left(\mathbb{P}\left(S^R_{(d)}\right)\right)} \int_{\mathbb{P}\left(S^R_{(d)}\right)} (f) d\mathbb{P}\left(S^R_{(d)}\right)$$

be the average number of real roots of system of homogeneous polynomial equations with real coefficients, of degree $(d)$.

**Theorem 4.1** We have $A_{(d)} = \sqrt{D}$, where $D = d_1 \times \cdots \times d_n$ is the Bezout number, ie. The average number of real roots of a real homogeneous polynomial system is exactly the square root of the Bezout number $D$.

**Applications**

The following results are derived immediately from Theorem 4.1.

**Corollary 4.1** The special case $n=1$, or $(d) = d$ then $A_d = \sqrt{d}$, ie. the average number of real roots of a real homogeneous polynomial equation of degree $d$ is $\sqrt{d}$.

**Corollary 4.2** For $(d) = (d_1, d_2)$, let $(V,W)$ denote the intersection numbers of two algebraic curves. We define

$$A_{(d_1,d_2)} = \frac{1}{\text{Vol}\left(\mathbb{P}\left(S^R_{(d)}\right)\right)} \int_{\mathbb{P}\left(S^R_{(d)}\right)} (V,W) d\mathbb{P}\left(S^R_{(d)}\right)$$

be the average intersection numbers of two algebraic curves in real projective space. Then $A_{(d_1,d_2)} = \sqrt{d_1 d_2}$.

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References


